Int. J. Production Economics 137 (2012) 102–115

Contents lists available at SciVerse ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Lot-sizing decisions for deteriorating items with two warehouses under an order-size-dependent trade credit

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ARTICLE INFO

Article history:
Received 18 September 2008
Accepted 11 January 2012
Available online 1 February 2012

Keywords:
Inventory
Deteriorating items
Two-warehouses
Trade credit

ABSTRACT

This study attempts to determine economic order quantity for deteriorating items with two-storage facilities (one is an owned warehouse and the other is a rented warehouse) where trade credit is linked to order quantity. As assumed herein, payment delays depend on the quantity ordered, when the order quantity is less than that at which a payment delay is permitted, the payment for the items must be made immediately. Otherwise, the fixed trade credit period is permitted. Furthermore, if the order quantity exceeds the owned warehouse capacity, it will be necessary to rent a warehouse which results in an additional rental cost. Otherwise, renting a warehouse is unnecessary. The problem discussed in this study involves how retailers decide whether to rent an additional warehouse to hold more items and thus obtain a trade credit period. First, a deterministic inventory model is developed for deteriorating items under the above situation. Second, this study demonstrates that the total cost function per unit time is convex via a rigorous proof. Third, five theorems are developed to optimize the replenishment cycle time and the order lot-size. Finally, numerical examples are used to illustrate these theorems and sensitivity analysis of the optimal solution with respect to the parameters of the system is carried out and some important managerial insights are obtained.

1. Introduction

In recent decades, many studies have examined the problem of managing deteriorating items including medicines, volatile liquids, blood banks, foodstuffs and electronic components. Raafat (1991) presented a complete survey of the inventory literature on deteriorating inventory models. Moreover, Ghare and Schrader (1963) the first proponents proposed for developing a revised form of the EOQ model that assumed exponential decay. Covert and Philip (1973) then extended this model to consider the Weibull distribution deterioration. Other noteable works in this include those of Dave and Patel (1981), Sachan (1984), Hariga (1996) and their references. Recently, Goyal and Giri (2001) presented a review of the inventory literature published on deteriorating items since the early 1990s.

The conventionally adopted EOQ model assumes that the retailer must pay to purchase the item immediately upon receiving it from a supplier. However, such an assumption does not necessarily reflect the scenario in the real world. In fact, suppliers generally allow retailers' access to forward financing to increase demand or decrease inventory. This means that the supplier permits a trade credit period for the settlement of payment. The effect of the trade credit on the optimal inventory model has been examined in various studies. Goyal (1985) established an inventory model under permissible delay in payments. Shah (1993a,b) designed EOQ models for perishable items where payment delay is permissible. Other noteable works on this area were by Chand and Ward (1987), Aggarwal and Jaggi (1995), Chung and Liao (2006), Jamal et al. (2000), Chung (1998), Daellenbach (1986), Shinn (1997), Shinn and Hwang (2003), Liao (2007a,b), Zaid (2011), Ruo et al. (2011), Musa and Sani (2011), Tsao and Sheen (2012) and others. In fact, a key finding of these studies was that EOQ is independent of trade credit. Chung and Liao (2004) and Chang et al. (2003) considered the deteriorating items given the conditions of an order-size-dependent trade credit.

All the aforementioned inventory models implicitly assumed that the retailer owns a single warehouse with unlimited capacity. However, in more practical terms, any warehouse has a limited capacity. On the other hand, due to some reasons such as an attracted price discount for bulk purchase, the order costs higher than one using rented warehouse, and so on, inventory managers usually are attracted to hold more items than can be stored in an owned

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doi:10.1016/j.ijpe.2012.01.020

warehouse. From this perspective, the two warehouse inventory models recently have been considered by various authors. This kind of system was first proposed by Hartely (1976). Sarma (1983) designed a deterministic inventory model with infinite replenishment rate and two storage levels. Furthermore, Murdeshwar and Sathe (1983) extended the case to incorporate finite replenishment rate. Other researchers that have studied in this area include Goswami and Chaudhuri (1992), Bhunia and Maiti (1998), Sarma (1987), Pakkala and Achary (1992a, 1992b), Benkherout (1997), Zhou (1998), Yang (2004) and Zhou and Yang (2005).

Due to the factors mentioned above, Chung and Huang (2006) considered a two-warehouse inventory problem for deteriorating item with limited storage space under permissible delay in payments. However, in certain practical situations, trade credits can be applied as an alternative to price discounts to order more quantities. Consequently, an important problem associated with inventory maintenance is deciding whether to rent an additional warehouse to hold more items to obtain a trade credit period.

Based on the above arguments, this study incorporates both Chung and Huang (2006) and Chung and Liao (2004) under above conditions. This study considers payment delay to depend on order quantity where the order quantity is less than that at which delayed payment is permitted, meaning payment must be made immediately. Otherwise, the fixed trade credit period is permitted. Additionally, if the order quantity exceeds owned warehouse capacity it becomes necessary to rent a warehouse which results in an additional rental cost. Given this marketing situation, this study develops a deterministic inventory model for deteriorating items with two warehouses (one is OW and the other is RW) and where trade credit is linked to order quantity. This study then demonstrates easy-to-use theorems to identify the optimal replenishment cycle time and the optimal order lot-size to minimize. Numerical examples are used to illustrate all of the study theorems and revealed the decision whether to rent an additional warehouse. Finally, sensitivity analysis of the optimal solution with respect to the parameters of the system is carried out and some important managerial insights are obtained.

2. Notations and assumptions

We adopt the following notations for the model to be discussed:

Notations

C	unit purchase cost
S	ordering cost
A	rental cost for renting an additional warehouse
M	credit period set by the supplier
h	unit stock holding cost for item in OW (excluding capital opportunity cost)
k	unit stock holding cost for item in RW (excluding capital opportunity cost)
R	capital opportunity cost (as a percentage)
I	earned interest rate (as a percentage)
Q	order size
T	replenishment cycle time
D	annual demand rate
λ	a constant deterioration rate
$\overline{W}$	quantity at which the delay in payments is permitted
W	the storage capacity of OW
t_W	the time that inventory level reduce to W

$$T_{\overline{W}} = \frac{1}{\lambda}\ln\left(\frac{\lambda}{D}\overline{W}+1\right), \quad T_a = \frac{1}{\lambda}\ln\left(\frac{\lambda}{D}W+1\right)$$

In addition, we adopt the following assumptions for the model to be discussed:

(1) Replenishments are instantaneous with a known and constant lead time.
(2) No shortages are allowed.
(3) The demand rate is known with certainty and is uniform.
(4) The supplier proposes a certain credit period in paying for purchasing cost and the sales revenue generated during the credit period is deposited in an interest bearing account with rate I. At the end of the period, the credit is settled and the retailer starts paying the capital opportunity cost for the items in stock with rate $R (R \geq I)$.
(5) The daily expenses of the system can be overcome from the difference between retail price and unit cost.
(6) The time to deterioration of each item follows an exponential distribution with parameter λ, and the deteriorated units are not replaced.
(7) If $Q < \overline{W}$, the delay in payments is not permitted. Otherwise, certain fixed trade credit period M is permitted.
(8) The owned warehouse (OW) has a fixed capacity of W units and the rented warehouse (RW) has unlimited capacity.
(9) The items of OW are consumed only after consuming the items kept in RW.
(10) The time of transporting items from RW to OW is ignored.

3. Development of the mathematical model

Let $Q(t)$ denote the system inventory level at time t, $(0 \leq t \leq T)$, the inventory level decreases according to demand as well as deterioration simultaneously. The change in inventory level can be represented using the following differential equation:

$$\frac{dQ(t)}{dt} + \lambda Q(t) = -D, \quad 0 \leq t \leq T \tag{1}$$

with the boundary condition $Q(T)=0$. The solution of Eq. (1) is

$$Q(t) = \frac{D}{\lambda}(e^{\lambda(T-t)}-1), \quad 0 \leq t \leq T \tag{2}$$

Additionally, the order quantity for each replenishment cycle is

$$Q = Q(0) = \frac{D}{\lambda}(e^{\lambda T}-1) \tag{3}$$

and the number of units that deteriorate per cycle is

$$Q - DT = \frac{D}{\lambda}(e^{\lambda T} - \lambda T - 1) \tag{4}$$

This study observes that if order quantity $Q \leq \overline{W}$, the payment must be made immediately. Otherwise, the retailer will get a certain credit period, M. Clearly, the inequality $Q \leq \overline{W}$ holds if and only if $T \leq (1/\lambda)\ln((\lambda/D)\overline{W}+1) = T_{\overline{W}}$. Additionally, this study assumes that the retailer owns a warehouse with a fixed capacity of W units, meaning that any quantity exceeding this must be stored in a rented warehouse. From this perspective, this study observes that if the order quantity $Q \leq W$, renting a warehouse is unnecessary. Otherwise, W units of items are stored in the OW and the remainder are dispatched in the RW. Subsequently, the inequality $Q \leq W$ holds if and only if $T \leq (1/\lambda)\ln((\lambda/D)W+1) = T_a$. Herein, total annual variable cost function can be expressed as follows:

$$\begin{aligned} TVC(T) = {} & \textit{ordering cost} + \textit{purchasing cost} + \textit{deteriorating cost} \\ & + \textit{stock-holding cost in RW} + \textit{stock-holding cost in OW} \\ & + \textit{capital opportunity cost} + \textit{rental cost in RW}. \end{aligned}$$

There are two possible cases (Case (A): $T_{\overline{W}} < M$) and (Case (B): $T_{\overline{W}} \geq M$) that arise and we discuss each case in detail:

(A) Suppose that $T_{\overline{W}} < M$

(a) Annual ordering cost$=S/T$
(b) Annual purchasing cost$=(CQ)/(T)=(CD)/(\lambda T)(e^{\lambda T}-1)$
(c) Annual deteriorating cost$=(CD)/(\lambda T)(e^{\lambda T}-\lambda T-1)$
(d) Annual rental cost and stock holding cost in RW:

Case (d1): $T \leq T_a$

In this case, it does not need to rent any warehouse and so no rental cost and stock holding cost for items in RW.

Case (d2): $T > T_a$

Annual rental cost in RW$=(A)/(T)$

$$\text{Annual stock holding cost in RW} = \frac{k}{T}\int_0^{t_W}(Q(t)-W)dt$$
$$= \frac{k}{\lambda^2 T}[D(e^{\lambda T}-e^{\lambda T_a})-(D\lambda+\lambda^2 W)(T-T_a)]$$

(e) Annual Stocking cost in OW:

Case (e1): $T \leq T_a$

$$\text{Annual stock holding cost in OW} = \frac{h}{T}\int_0^T Q(t)dt = \frac{hD}{\lambda^2 T}(e^{\lambda T}-\lambda T-1)$$

Case (e2): $T > T_a$

Annual stock holding cost in OW
$$= \frac{h}{T}\left(Wt_W + \int_{t_W}^T Q(t)dt\right) = \frac{h}{\lambda^2 T}[D(e^{\lambda T_a}-\lambda T_a-1)+\lambda^2 W(T-T_a)]$$

(f) Annual capital opportunity cost:

Case (f1): $0 < T < T_{\overline{W}}$

Annual capital opportunity cost
$$= \frac{CR}{T}\int_0^T Q(t)dt = \frac{CRD}{\lambda^2 T}(e^{\lambda T}-\lambda T-1)$$

Case (f2): $T_{\overline{W}} \leq T < M$

Annual capital opportunity cost$=(CIDT)/(2)-CIDM$

Case (f3): $M \leq T$

Annual capital opportunity cost$=(CRD)/(\lambda^2 T)[e^{\lambda(T-M-}\lambda(T-M)-1]-(CIDM^2)/(2T)$

Next, after collecting the values of the individual term under different circumstances, this study considers the following cases when $T_{\overline{W}} < M$:

Case (I): $T_a < T_{\overline{W}} < M$

In this case, the total annual inventory cost is obtained as follows:

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_a \\ TVC_2(T) & \text{if} \quad T_a \leq T < T_{\overline{W}} \\ TVC_3(T) & \text{if} \quad T_{\overline{W}} \leq T < M \\ TVC_4(T) & \text{if} \quad M \leq T \end{cases} \tag{5a–d}$$

where

$$TVC_1(T) = \frac{S}{T} + \frac{CD}{\lambda T}(e^{\lambda T}-1) + \frac{D(\lambda C+h+CR)}{\lambda^2 T}(e^{\lambda T}-\lambda T-1) \tag{6}$$

$$TVC_2(T) = \frac{(S+A)}{T} + \frac{CD}{\lambda T}(e^{\lambda T}-1) + \frac{D(\lambda C+k+CR)}{\lambda^2 T}(e^{\lambda T}-\lambda T-1) - \frac{(k-h)}{\lambda^2 T}[D(e^{\lambda T_a}-\lambda T_a-1)+\lambda^2 W(T-T_a)] \tag{7}$$

$$TVC_3(T) = \frac{(S+A)}{T} + \frac{CD}{\lambda T}(e^{\lambda T}-1) + \frac{D(\lambda C+k)}{\lambda^2 T}(e^{\lambda T}-\lambda T-1) - \frac{(k-h)}{\lambda^2 T}[D(e^{\lambda T_a}-\lambda T_a-1)+\lambda^2 W(T-T_a)] - DCI\left(M-\frac{T}{2}\right) \tag{8}$$

and

$$TVC_4(T) = \frac{(S+A)}{T} + \frac{CD}{\lambda T}(e^{\lambda T}-1) + \frac{D(\lambda C+k)}{\lambda^2 T}(e^{\lambda T}-\lambda T-1) - \frac{(k-h)}{\lambda^2 T}[D(e^{\lambda T_a}-\lambda T_a-1)+\lambda^2 W(T-T_a)] + \frac{CRD}{\lambda^2 T}[e^{\lambda(T-M)}-\lambda(T-M)-1] - \frac{CIDM^2}{2T} \tag{9}$$

Case (II): $T_{\overline{W}} \leq T_a < M$

In this case, the total inventory cost is given by

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_{\overline{W}} \\ TVC_5(T) & \text{if} \quad T_{\overline{W}} \leq T < T_a \\ TVC_3(T) & \text{if} \quad T_a \leq T < M \\ TVC_4(T) & \text{if} \quad M \leq T \end{cases} \tag{10a–d}$$

where

$$TVC_5(T) = \frac{S}{T} + \frac{CD}{\lambda T}(e^{\lambda T}-1) + \frac{D(\lambda C+h)}{\lambda^2 T}(e^{\lambda T}-\lambda T-1) - DCI\left(M-\frac{T}{2}\right) \tag{11}$$

Case (III): $T_{\overline{W}} < M < T_a$

In this case, the total inventory cost is given by

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_{\overline{W}} \\ TVC_5(T) & \text{if} \quad T_{\overline{W}} \leq T < M \\ TVC_6(T) & \text{if} \quad M \leq T < T_a \\ TVC_4(T) & \text{if} \quad T_a \leq T \end{cases} \tag{12a–d}$$

where

$$TVC_6(T) = \frac{S}{T} + \frac{CD}{\lambda T}(e^{\lambda T}-1) + \frac{D(\lambda C+h)}{\lambda^2 T}(e^{\lambda T}-\lambda T-1) + \frac{CRD}{\lambda^2 T}(e^{\lambda(T-M)}-\lambda(T-M)-1) - \frac{CIDM^2}{2T} \tag{13}$$

(B) Suppose that $M \leq T_{\overline{W}}$

Similarly, three cases need to be discussed when $M \leq T_{\overline{W}}$:

Case (I): $T_a < M < T_{\overline{W}}$

In this case, $TVC(T)$ can be expressed as follows:

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_a \\ TVC_2(T) & \text{if} \quad T_a \leq T < T_{\overline{W}} \\ TVC_4(T) & \text{if} \quad T_{\overline{W}} \leq T \end{cases} \tag{14a–c}$$

Case (II): $M < T_a < T_{\overline{W}}$

In this case, the annual total variable cost function is the same in Case (I).

Case (III): $M < T_{\overline{W}} < T_a$

In this case, $TVC(T)$ can be expressed as follows:

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_{\overline{W}} \\ TVC_6(T) & \text{if} \quad T_{\overline{W}} \leq T < T_a \\ TVC_4(T) & \text{if} \quad T_a \leq T \end{cases} \tag{15a–c}$$

Further, we obtain the following results.

Lemma 1.

(1) *When $M > T_{\overline{W}}$, TVC(T) is continuous except when $T = T_{\overline{W}}$ and $T = T_a$.*

(2) *When* $M \le T_{\overline{W}}$, *TVC(T) is continuous except when* $T = T_{\overline{W}}$ *and* $T = T_a$.

Proof. The proof of Lemma 1 is given in Appendix A.

Most strikingly, to optimize the solutions to Eqs. (6)–(9), (11) and (13), it is necessary to study $TVC_i(T)$ (i=1, 2, 3, 4, 5 and 6), respectively. Furthermore, the first order derivative of $TVC_i(T)$ (i=1, 2, 3, 4, 5 and 6) is

$$TVC_1'(T) = -\frac{S}{T^2} + \frac{D(2\lambda C + h + CR)f(T)}{\lambda^2 T^2} \tag{16}$$

$$TVC_2'(T) = -\frac{(S+A)}{T^2} + \frac{D(2\lambda C + k + CR)f(T)}{\lambda^2 T^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T^2} \tag{17}$$

$$TVC_3'(T) = -\frac{(S+A)}{T^2} + \frac{D(2\lambda C + k)f(T)}{\lambda^2 T^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T^2} + \frac{CID}{2} \tag{18}$$

$$TVC_4'(T) = -\frac{(S+A)}{T^2} + \frac{D(2\lambda C + k)f(T)}{\lambda^2 T^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T^2} + \frac{CRD\,g(T)}{\lambda^2 T^2} + \frac{CIDM^2}{2T^2} \tag{19}$$

$$TVC_5'(T) = -\frac{S}{T^2} + \frac{D(2\lambda C + h)f(T)}{\lambda^2 T^2} + \frac{CID}{2} \tag{20}$$

and

$$TVC_6'(T) = -\frac{S}{T^2} + \frac{D(2\lambda C + h)f(T)}{\lambda^2 T^2} + \frac{CRD\,g(T)}{\lambda^2 T^2} + \frac{CIDM^2}{2T^2} \tag{21}$$

where

$$f(T) = \lambda T e^{\lambda T} - e^{\lambda T} + 1$$

and

$$g(T) = \lambda T e^{\lambda(T-M)} - e^{\lambda(T-M)} - \lambda M + 1$$

Then, we have the following result:

Lemma 2.

(1) $f(T) > 0$ *for* $T > 0$.
(2) $g(T) \ge 0$ *for* $T \ge M$.

Proof. The proof of Lemma 2 is given in Appendix B. □

Next, the second order derivative is

$$TVC_1''(T) = \frac{2S}{T^3} + \frac{D(2\lambda C + h + CR)h(T)}{\lambda^2 T^3} \tag{22}$$

$$TVC_2''(T) = \frac{2(S+A)}{T^3} + \frac{D(2\lambda C + k + CR)h(T)}{\lambda^2 T^3} + \frac{2D(k-h)f(T_a)}{\lambda^2 T^3} \tag{23}$$

$$TVC_3''(T) = \frac{2(S+A)}{T^3} + \frac{D(2\lambda C + k)h(T)}{\lambda^2 T^3} + \frac{2D(k-h)f(T_a)}{\lambda^2 T^3} \tag{24}$$

$$TVC_4''(T) = \frac{2(S+A)}{T^3} + \frac{D(2\lambda C + k)h(T)}{\lambda^2 T^3} + \frac{2D(k-h)f(T_a)}{\lambda^2 T^3} + \frac{CD\,k(T)}{\lambda^2 T^3} \tag{25}$$

$$TVC_5''(T) = \frac{2S}{T^3} + \frac{D(2\lambda C + h)h(T)}{\lambda^2 T^3} \tag{26}$$

and

$$TVC_6''(T) = \frac{2S}{T^3} + \frac{D(2\lambda C + h)h(T)}{\lambda^2 T^3} + \frac{CDk(T)}{\lambda^2 T^3} \tag{27}$$

where

$$h(T) = \lambda^2 T^2 e^{\lambda T} - 2\lambda T e^{\lambda T} + 2e^{\lambda T} - 2$$

and

$$k(T) = R(\lambda^2 T^2 e^{\lambda(T-M)} - 2\lambda T e^{\lambda(T-M)} + 2e^{\lambda(T-M)} + 2\lambda M - 2) - I\lambda^2 M^2$$

The main result obtained in this section is as follows:

Theorem 1.

(1) $TVC_1(T)$ *is convex for* $T > 0$.
(2) $TVC_2(T)$ *is convex for* $T > 0$.
(3) $TVC_3(T)$ *is convex for* $T > 0$.
(4) $TVC_4(T)$ *is convex for* $T > M$.
(5) $TVC_5(T)$ *is convex for* $T > 0$.
(6) $TVC_6(T)$ *is convex for* $T > M$.

Proof. The proof of Theorem 1 is given in Appendix C. □

As a matter of fact, from parts (1), (2), (3) and (5) of Theorem 1, this study finds that $TVC'_i(T)$ (i=1, 2, 3 and 5) is increasing on $(0,\infty)$. Use the rule of L'Hopital, $\lim_{T\to 0^+} TVC_i'(T) = -\infty$ and $\lim_{T\to\infty} TVC_i'(T) = \infty$ for i=1, 2, 3 and 5. Furthermore, the Intermediate Value Theorem yields that $TVC'_i(T)=0$ has a unique solution T_i^* (i=1, 2, 3 and 5) on $(0,\infty)$.

Meanwhile, parts (4) and (6) of Theorem 1 indicate that $TVC'_j(T)$ (j=4 and 6) is increasing on $[M,\infty)$. Obviously, $\lim_{T\to\infty} TVC_j'(T) = \infty$ (j=4 and 6). Next, supposing $TVC'_j(M)$ is negative (j=4 and 6), the Intermediate Value Theorem supports the existence of T_j^* (j=4 and 6). Otherwise, based on the convexity of $TVC_j(T)$ (j=4 and 6) on $[M,\infty)$, $TVC_j(T)$ (j=4 and 6) is increasing on $[M,\infty)$.

4. Determination of the optimal ordering policy when $T_{\overline{W}} < M$

This section develops simple, rapid and accurate decision rules for optimizing the replenishment cycle time when $T_{\overline{W}} < M$.

Case (I): $T_a < T_{\overline{W}} < M$

From Eq. (5 (a)–(d)), then

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_a \\ TVC_2(T) & \text{if} \quad T_a \le T < T_{\overline{W}} \\ TVC_3(T) & \text{if} \quad T_{\overline{W}} \le T < M \\ TVC_4(T) & \text{if} \quad M \le T \end{cases}$$

Furthermore, Eqs. (16)–(19) yield

$$TVC_1'(T_a) = -\frac{S}{T_a^2} + \frac{D(2\lambda C + h + CR)f(T_a)}{\lambda^2 T_a^2} \tag{28}$$

$$TVC_2'(T_a) = -\frac{(S+A)}{T_a^2} + \frac{D(2\lambda C + k + CR)f(T_a)}{\lambda^2 T_a^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T_a^2} \tag{29}$$

$$TVC_2'(T_{\overline{W}}) = -\frac{(S+A)}{T_{\overline{W}}^2} + \frac{D(2\lambda C + k + CR)f(T_{\overline{W}})}{\lambda^2 T_{\overline{W}}^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T_{\overline{W}}^2} \tag{30}$$

$$TVC_3'(T_{\overline{W}}) = -\frac{(S+A)}{T_{\overline{W}}^2} + \frac{D(2\lambda C + k)f(T_{\overline{W}})}{\lambda^2 T_{\overline{W}}^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T_{\overline{W}}^2} + \frac{CID}{2} \tag{31}$$

$$TVC_3'(M) = -\frac{(S+A)}{M^2} + \frac{D(2\lambda C + k)f(M)}{\lambda^2 M^2} - \frac{D(k-h)f(T_a)}{\lambda^2 M^2} + \frac{CID}{2} \tag{32}$$

and

$$TVC_4'(M) = -\frac{(S+A)}{M^2} + \frac{D(2\lambda C + k)f(M)}{\lambda^2 M^2} - \frac{D(k-h)f(T_a)}{\lambda^2 M^2} + \frac{CID}{2} \tag{33}$$

It is easy to check that $TVC'_1(T_a) > TVC'_2(T_a)$ and $TVC'_3(M) = TVC'_4(M)$.

Eqs. (17) and (18) yield

$$TVC'_2(T) - TVC'_3(T) = \frac{CRD}{\lambda^2 T^2} f(T) - \frac{CID}{2} > \frac{CRD}{2\lambda^2 T^2}[2f(T) - \lambda^2 T^2]$$

Fortunately, Lemma 3 in Chung and Liao (2004) implies that $2 \cdot f(T) - \lambda^2 T^2$ is positive for $T > 0$. Based on this, it is obtained that $TVC'_2(T) > TVC'_3(T)$ for $T > 0$. Given $T = T_{\overline{W}}$, then $TVC'_2(T_{\overline{W}}) > TVC'_3(T_{\overline{W}})$. Moreover, since $TVC''_2(T)$ and $TVC''_3(T)$ are positive for $T > 0$, then $TVC'_2(T)$ and $TVC'_3(T)$ are increasing on $T > 0$ and thus $TVC'_2(T_a) < TVC'_2(T_{\overline{W}})$ and $TVC'_3(T_{\overline{W}}) < TVC'_3(M)$. Next, for notational convenience, the following are allowed:

$\Delta_1 = TVC'_1(T_a)$, $\Delta_1^* = TVC'_2(T_a)$, $\Delta_2 = TVC'_2(T_{\overline{W}})$, $\Delta_3 = TVC'_3(T_{\overline{W}})$, $\Delta_4 = TVC'_3(M) = TVC'_4(M)$.

Therefore, $\Delta_1^* < \Delta_1$, $\Delta_1^* < \Delta_2$, $\Delta_3 < \Delta_2$ and $\Delta_3 < \Delta_4$. Combining the above situations, the following theorem is obtained:

Theorem 2.

(1) *If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$, $\Delta_3 < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_1(T_a), TVC_4(T_4^*)\}$ and $T^* = T_a$ or T_4^* associated with the least cost.*

(2) *If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$, $\Delta_3 < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_a), TVC_3(T_3^*)\}$ and $T^* = T_a$ or T_3^* associated with the least cost.*

(3) *If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 \ge 0$, $\Delta_3 < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_1(T_a), TVC_4(T_4^*)\}$ and $T^* = T_a$ or T_4^* associated with the least cost.*

(4) *If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 \ge 0$, $\Delta_3 < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_a), TVC_3(T_3^*)\}$ and $T^* = T_a$ or T_3^* associated with the least cost.*

(5) *If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 \ge 0$, $\Delta_3 \ge 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_a), TVC_2(T_2^*), TVC_3(T_{\overline{W}})\}$ and $T^* = T_a$, T_2^* or $T_{\overline{W}}$ associated with the least cost.*

(6) *If $\Delta_1 \ge 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$, $\Delta_3 < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_4^*)\}$ and $T^* = T_1^*$ or T_4^* associated with the least cost.*

(7) *If $\Delta_1 \ge 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$, $\Delta_3 < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_3(T_3^*)\}$ and $T^* = T_1^*$ or T_3^* associated with the least cost.*

(8) *If $\Delta_1 \ge 0$, $\Delta_1^* < 0$, $\Delta_2 \ge 0$, $\Delta_3 < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_4^*)\}$ and $T^* = T_1^*$ or T_4^* associated with the least cost.*

(9) *If $\Delta_1 \ge 0$, $\Delta_1^* < 0$, $\Delta_2 \ge 0$, $\Delta_3 < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_3(T_3^*)\}$ and $T^* = T_1^*$ or T_3^* associated with the least cost.*

(10) *If $\Delta_1 \ge 0$, $\Delta_1^* < 0$, $\Delta_2 \ge 0$, $\Delta_3 \ge 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_2(T_2^*), TVC_3(T_{\overline{W}})\}$ and $T^* = T_1^*$, T_2^* or $T_{\overline{W}}$ associated with the least cost.*

(11) *If $\Delta_1 \ge 0$, $\Delta_1^* \ge 0$, $\Delta_2 \ge 0$, $\Delta_3 < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_4^*)\}$ and $T^* = T_1^*$ or T_4^* associated with the least cost.*

(12) *If $\Delta_1 \ge 0$, $\Delta_1^* \ge 0$, $\Delta_2 \ge 0$, $\Delta_3 < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_3(T_3^*)\}$ and $T^* = T_1^*$ or T_3^* associated with the least cost.*

(13) *If $\Delta_1 \ge 0$, $\Delta_1^* \ge 0$, $\Delta_2 \ge 0$, $\Delta_3 \ge 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_3(T_{\overline{W}})\}$ and $T^* = T_1^*$ or $T_{\overline{W}}$ associated with the least cost.*

Proof. The proof of Theorem 2 is given in Appendix D.

Case (II): $T_{\overline{W}} < T_a < M$

From Eq. (10(a)–(d)), then

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_{\overline{W}} \\ TVC_5(T) & \text{if} \quad T_{\overline{W}} \le T < T_a \\ TVC_3(T) & \text{if} \quad T_a \le T < M \\ TVC_4(T) & \text{if} \quad M \le T \end{cases}$$

Eqs. (16), (20) and (18) yield that

$$TVC'_1(T_{\overline{W}}) = -\frac{S}{T_{\overline{W}}^2} + \frac{D(2\lambda C + h + CR)f(T_{\overline{W}})}{\lambda^2 T_{\overline{W}}^2} \quad (34)$$

$$TVC'_5(T_{\overline{W}}) = -\frac{S}{T_{\overline{W}}^2} + \frac{D(2\lambda C + h)f(T_{\overline{W}})}{\lambda^2 T_{\overline{W}}^2} + \frac{CID}{2} \quad (35)$$

$$TVC'_5(T_a) = -\frac{S}{T_a^2} + \frac{D(2\lambda C + h)f(T_a)}{\lambda^2 T_a^2} + \frac{CID}{2} \quad (36)$$

and

$$TVC'_3(T_a) = -\frac{(S+A)}{T_a^2} + \frac{D(2\lambda C + k)f(T_a)}{\lambda^2 T_a^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T_a^2} + \frac{CID}{2} \quad (37)$$

Clearly, $TVC'_5(T_a) > TVC'_3(T_a)$. Additionally, the fact that $TVC''_5(T)$ is positive for $T > 0$ implies that $TVC'_5(T)$ is increasing on $T > 0$ and thus $TVC'_5(T_{\overline{W}}) < TVC'_5(T_a)$.

Eqs. (16) and (20) yield

$$TVC'_1(T) - TVC'_5(T) = \frac{CRD}{\lambda^2 T^2} f(T) - \frac{CID}{2}$$

Based on the above, $TVC'_1(T) > TVC'_5(T)$ for $T > 0$. Given $T = T_{\overline{W}}$, then $TVC'_1(T_{\overline{W}}) > TVC'_5(T_{\overline{W}})$. Next, for notational convenience, let $\overline{\Delta}_1 = TVC'_1(T_{\overline{W}})$, $\overline{\Delta}_2 = TVC'_5(T_{\overline{W}})$, $\overline{\Delta}_3 = TVC'_5(T_a)$and $\overline{\Delta}_3^* = TVC'_3(T_a)$. Therefore, $\overline{\Delta}_2 < \overline{\Delta}_1$, $\overline{\Delta}_2 < \overline{\Delta}_3$, $\overline{\Delta}_3^* < \overline{\Delta}_3$ and $\overline{\Delta}_3^* < \Delta_4$. Combining the above situations, the following theorem is obtained:

Theorem 3.

(1) *If $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 < 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_5(T_a), TVC_4(T_4^*)\}$ and $T^* = T_a$ or T_4^* associated with the least cost.*

(2) *If $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 < 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_5(T_a), TVC_3(T_3^*)\}$ and $T^* = T_a$ or T_3^* associated with the least cost.*

(3) *If $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 \ge 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_5(T_5^*), TVC_4(T_4^*)\}$ and $T^* = T_5^*$ or T_4^* associated with the least cost.*

(4) *If $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 \ge 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_5(T_5^*), TVC_3(T_3^*)\}$ and $T^* = T_5^*$ or T_3^* associated with the least cost.*

(5) *If $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 \ge 0$, $\overline{\Delta}_3^* \ge 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = TVC_5(T_5^*)$, and $T^* = T_5^*$.*

(6) *If $\overline{\Delta}_1 \ge 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 < 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_5(T_a), TVC_4(T_4^*)\}$ and $T^* = T_a$ or T_4^* associated with the least cost.*

(7) *If $\overline{\Delta}_1 \ge 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 < 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_5(T_a), TVC_3(T_3^*)\}$ and $T^* = T_a$ or T_3^* associated with the least cost.*

(8) *If $\overline{\Delta}_1 \ge 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 \ge 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 < 0$, then $TVC(T^*) = \min\{TVC_5(T_5^*), TVC_4(T_4^*)\}$ and $T^* = T_5^*$ or T_4^* associated with the least cost.*

(9) *If $\overline{\Delta}_1 \ge 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 \ge 0$, $\overline{\Delta}_3^* < 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = \min\{TVC_5(T_5^*), TVC_3(T_3^*)\}$ and $T^* = T_5^*$ or T_3^* associated with the least cost.*

(10) *If $\overline{\Delta}_1 \ge 0$, $\overline{\Delta}_2 < 0$, $\overline{\Delta}_3 \ge 0$, $\overline{\Delta}_3^* \ge 0$ and $\Delta_4 \ge 0$, then $TVC(T^*) = TVC_5(T_5^*)$, and $T^* = T_5^*$.*

(11) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 \geq 0$, $\overline{\Delta}_3 \geq 0$, $\overline{\Delta}_3^* < 0$ *and* $\Delta_4 < 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_5(T_{\overline{W}}), TVC_4(T_4^*)\}$ *and* $T^* = T_1^*$, $T_{\overline{W}}$ *or* T_4^* *associated with the least cost.*
(12) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 \geq 0$, $\overline{\Delta}_3 \geq 0$, $\overline{\Delta}_3^* < 0$ *and* $\Delta_4 \geq 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_5(T_{\overline{W}}), TVC_3(T_3^*)\}$ *and* $T^* = T_1^*$, $T_{\overline{W}}$ *or* T_3^* *associated with the least cost.*
(13) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 \geq 0$, $\overline{\Delta}_3 \geq 0$, $\overline{\Delta}_3^* \geq 0$ *and* $\Delta_4 \geq 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_5(T_{\overline{W}})\}$ *and* $T^* = T_1^*$ *or* $T_{\overline{W}}$ *associated with the least cost.*

Proof. The proof of Theorem 3 is similar to the proof of Theorem 2. □

Case (III): $T_{\overline{W}} < M < T_a$
Eq. (12)(a–d) implies that

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_{\overline{W}} \\ TVC_5(T) & \text{if} \quad T_{\overline{W}} \leq T < M \\ TVC_6(T) & \text{if} \quad M \leq T < T_a \\ TVC_4(T) & \text{if} \quad T_a \leq T \end{cases}$$

Eqs. (20), (21), and (19) yield that

$$TVC_5'(M) = -\frac{S}{M^2} + \frac{D(2\lambda C + h)f(M)}{\lambda^2 M^2} + \frac{CID}{2} \tag{38}$$

$$TVC_6'(M) = -\frac{S}{M^2} + \frac{D(2\lambda C + h)f(M)}{\lambda^2 M^2} + \frac{CRD\, g(M)}{\lambda^2 M^2} + \frac{CID}{2} \tag{39}$$

$$TVC_6'(T_a) = -\frac{S}{T_a^2} + \frac{D(2\lambda C + h)f(T_a)}{\lambda^2 T_a^2} + \frac{CRD\, g(T_a)}{\lambda^2 T_a^2} + \frac{CIDM^2}{2T_a^2} \tag{40}$$

and

$$TVC_4'(T_a) = -\frac{(S+A)}{T_a^2} + \frac{D(2\lambda C + k)f(T_a)}{\lambda^2 T_a^2} - \frac{D(k-h)f(T_a)}{\lambda^2 T_a^2} + \frac{CRD\, g(T_a)}{\lambda^2 {T_a}^2} + \frac{CIDM^2}{2T_a^2} \tag{41}$$

Obviously, $TVC'_5(M) = TVC'_6(M)$ and $TVC'_6(T_a) > TVC'_4(T_a)$. Additionally, the fact that $TVC''_6(T)$ is positive when $T > 0$ implies that $TVC'_6(T)$ is increasing for $T > 0$, and thus $TVC'_6(M) < TVC'_6(T_a)$. To simplify the notation, the above can be rearranged as $\Delta_3^* = TVC_5'(M) = TVC_6'(M)$, $\Delta_4^* = TVC'_6(T_a)$ and $\overline{\Delta}_4^* = TVC_4'(T_a)$. Therefore, $\overline{\Delta}_2 < \Delta_3^*$, $\Delta_3^* < \Delta_4^*$ and $\overline{\Delta}_4^* < \Delta_4^*$. Combining the above situations yields the following theorem:

Theorem 4.

(1) *If* $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* < 0$, $\Delta_4^* < 0$ *and* $\overline{\Delta}_4^* < 0$, *then* $TVC(T^*) = \min\{TVC_6(T_a), TVC_4(T_4^*)\}$ *and* $T^* = T_a$ or T_4^* *associated with the least cost.*
(2) *If* $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* < 0$, $\Delta_4^* \geq 0$ and $\overline{\Delta}_4^* < 0$, then $TVC(T^*) = \min\{TVC_6(T_6^*), TVC_4(T_4^*)\}$. and $T^* = T_6^*$ *or* T_4^* *associated with the least cost.*
(3) *If* $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* < 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* \geq 0$, *then* $TVC(T^*) = TVC_6(T_6^*)$, *and* $T^* = T_6^*$.
(4) *If* $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* \geq 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* < 0$, *then* $TVC(T^*) = \min\{TVC_5(T_5^*), TVC_4(T_4^*)\}$ *and* $T^* = T_5^*$ *or* T_4^* *associated with the least cost.*
(5) *If* $\overline{\Delta}_1 < 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* \geq 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* \geq 0$, *then* $TVC(T^*) = TVC_5(T_5^*)$, *and* $T^* = T_5^*$.
(6) If $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* < 0$, $\Delta_4^* < 0$ *and* $\overline{\Delta}_4^* < 0$, *then* $TVC(T^*) = \min\{TVC_6(T_a), TVC_4(T_4^*)\}$ *and* $T^* = T_a$ or T_4^* *associated with the least cost.*
(7) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* < 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* < 0$, *then* $TVC(T^*) = \min\{TVC_6(T_6^*), TVC_4(T_4^*)\}$ *and* $T^* = T_6^*$ *or* T_4^* *associated with the least cost.*
(8) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* < 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* \geq 0$, *then* $TVC(T^*) = TVC_6(T_6^*)$, *and* $T^* = T_6^*$
(9) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* \geq 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* < 0$, *then* $TVC(T^*) = \min\{TVC_5(T_5^*), TVC_4(T_4^*)\}$ *and* $T^* = T_5^*$ *or* T_4^* *associated with the least cost.*
(10) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 < 0$, $\Delta_3^* \geq 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* \geq 0$, *then* $TVC(T^*) = TVC_5(T_5^*)$, *and* $T^* = T_5^*$.
(11) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 \geq 0$, $\Delta_3^* \geq 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* < 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_5(T_{\overline{W}}), TVC_4(T_4^*)\}$ *and* $T^* = T_1^*$, $T_{\overline{W}}$ *or* T_4^* *associated with the least cost.*
(12) *If* $\overline{\Delta}_1 \geq 0$, $\overline{\Delta}_2 \geq 0$, $\Delta_3^* \geq 0$, $\Delta_4^* \geq 0$ *and* $\overline{\Delta}_4^* \geq 0$, *then* $TVC(T^*) = \min\{TVC(T_1^*), TVC_5(T_{\overline{W}})\}$ *and* $T^* = T_1^*$ *or* $T_{\overline{W}}$ *associated with the least cost.*

Proof. The proof of Theorem 4 is similar to the proof of Theorem 2. □

5. Determination of the optimal ordering policy when $M \leq T_{\overline{W}}$

When $M \leq T_{\overline{W}}$, three cases require discussion:
Case (I): $T_a < M < T_{\overline{W}}$
In this case, $TVC(T)$ can be expressed as follows:

$$TVC(T) = \begin{cases} TVC_1(T) & \text{if} \quad 0 < T < T_a \\ TVC_2(T) & \text{if} \quad T_a \leq T < T_{\overline{W}} \\ TVC_4(T) & \text{if} \quad T_{\overline{W}} \leq T \end{cases} \tag{34a–c}$$

Based on Eqs. (17) and (19), the following can be obtained:

$$TVC_2'(T) - TVC_4'(T) = \frac{CRD}{\lambda^2 T^2}(f(T) - g(T)) - \frac{DCIM^2}{2T^2}$$
$$> \frac{CRD}{\lambda^2 T^2}\left[f(T) - g(T) - \frac{\lambda^2 M^2}{2}\right]$$

Lemma 4 in Chung and Liao (2004) implies that $f(T) - g(T) - (\lambda^2 M^2)/(2)$ is positive for $T \geq M$, yielding $TVC'_2(T) > TVC'_4(T)$ if $T \geq M$. Given $T = T_{\overline{W}}$, then $TVC_2'(T_{\overline{W}}) > TVC_4'(T_{\overline{W}})$. Likewise, $\tilde{\Delta}_4 = TVC_4'(T_{\overline{W}})$. Consequently, $\Delta_2 > \tilde{\Delta}_4$. The following result is thus obtained:

Theorem 5.

(1) *If* $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$ *and* $\tilde{\Delta}_4 < 0$, *then* $TVC(T^*) = \min\{TVC_1(T_a), TVC_4(T_4^*)\}$ *and* $T^* = T_a$ *or* T_4^* *associated with the least cost.*
(2) *If* $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ *and* $\tilde{\Delta}_4 < 0$, *then* $TVC(T^*) = \min\{TVC_1(T_a), TVC_2(T_2^*), TVC_4(T_4^*)\}$ *and* $T^* = T_a$, T_2^* *or* T_4^* *associated with the least cost.*
(3) *If* $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ *and* $\tilde{\Delta}_4 \geq 0$, *then* $TVC(T^*) = \min\{TVC_1(T_a), TVC_2(T_2^*), TVC_4(T_{\overline{W}})\}$ *and* $T^* = T_a$, T_2^* *or* $T_{\overline{W}}$ *associated with the least cost.*
(4) *If* $\Delta_1 \geq 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$ *and* $\tilde{\Delta}_4 < 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_4^*)\}$ *and* $T^* = T_1^*$ *or* T_4^* *associated with the least cost.*
(5) *If* $\Delta_1 \geq 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ *and* $\tilde{\Delta}_4 < 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_2(T_2^*), TVC_4(T_4^*)\}$ *and* $T^* = T_1^*$, T_2^* *or* T_4^* *associated with the least cost.*
(6) *If* $\Delta_1 \geq 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ *and* $\tilde{\Delta}_4 \geq 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_2(T_2^*), TVC_4(T_{\overline{W}})\}$ *and* $T^* = T_1^*$, T_2^* *or* $T_{\overline{W}}$ *associated with the least cost.*
(7) *If* $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$, $\Delta_2 \geq 0$ *and* $\tilde{\Delta}_4 < 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_4^*)\}$ *and* $T^* = T_1^*$ *or* T_4^* *associated with the least cost.*
(8) *If* $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$, $\Delta_2 \geq 0$ *and* $\tilde{\Delta}_4 \geq 0$, *then* $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_{\overline{W}})\}$ *and* $T^* = T_1^*$ *or* $T_{\overline{W}}$ *associated with the least cost.*

Proof. The proof of Theorem 5 is given in Appendix E. □

Case (II): $M < T_a < T_{\overline{W}}$
In this case, the optimal ordering policy is the same as in Case (I).

Case (III): $M<T_{\overline{W}}<T_a$

In this case, $TVC(T)$ can be expressed as follows:

$$TVC(T)=\begin{cases} TVC_1(T) & \text{if} \quad 0<T<T_{\overline{W}} \\ TVC_6(T) & \text{if} \quad T_{\overline{W}}\le T<T_a \\ TVC_4(T) & \text{if} \quad T_a\le T \end{cases} \qquad (35a\text{–}c)$$

Based on Eqs. (16) and (21), the following can be obtained:

$$TVC_1'(T)-TVC_6'(T)=\frac{DCR}{\lambda^2T^2}(f(T)-g(T))-\frac{CIDM^2}{2T^2}$$

As shown above, $TVC'_1(T)>TVC'_6(T)$ if $T\ge M$. Given $T=T_{\overline{W}}$, then $TVC_1'(T_{\overline{W}})>TVC_6'(T_{\overline{W}})$. Moreover, let $\Delta_2^*=TVC_6'(T_{\overline{W}})$, then $\overline{\Delta}_1>\Delta_2^*$, in which case the following result is obtained:

Theorem 6.

(1) *If* $\overline{\Delta}_1<0$, $\Delta_2^*<0$, $\Delta_4^*<0$ *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=\min\{TVC_6(T_a),TVC_4(T_4^*)\}$ *and* $T^*=T_a$ *or* T_4^* *associated with the least cost.*

(2) *If* $\overline{\Delta}_1<0$, $\Delta_2^*<0$, $\Delta_4^*\ge 0$ *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=\min\{TVC_6(T_6^*),TVC_4(T_4^*)\}$ *and* $T^*=T_a$ *or* T_4^* *associated with the least cost.*

(3) *If* $\overline{\Delta}_1<0$, $\Delta_2^*<0$, *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=TVC_6(T_6^*)$, *and* $T^*=T_6^*$ *and* $Q^*=D(e^{\lambda T_6^*-1})/\lambda$.

(4) *If* $\overline{\Delta}_1\ge 0$, $\Delta_2^*<0$, $\Delta_4^*<0$ *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_6(T_a),TVC_4(T_4^*)\}$ *and* $T^*=T_1^*$, T_a *or* T_4^* *associated with the least cost.*

(5) *If* $\overline{\Delta}_1\ge 0$, $\Delta_2^*<0$, $\Delta_4^*\ge 0$ *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_6(T_6^*),TVC_4(T_4^*)\}$ *and* $T^*=T_1^*$, T_6^* *or* T_4^* *associated with the least cost.*

(6) *If* $\overline{\Delta}_1\ge 0$, $\Delta_2^*<0$, $\Delta_4^*\ge 0$ *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_6(T_6^*)\}$ *and* $T^*=T_1^*$ *or* T_6^* *associated with the least cost.*

(7) *If* $\overline{\Delta}_1\ge 0$, $\Delta_2^*\ge 0$, $\Delta_4^*\ge 0$ *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_6(T_{\overline{W}}),TVC_4(T_4^*)\}$ *and* $T^*=T_1^*$, $T_{\overline{W}}$ *or* T_4^* *associated with the least cost.*

(8) *If* $\overline{\Delta}_1\ge 0$, $\Delta_2^*\ge 0$, $\Delta_4^*\ge 0$ *and* $\overline{\Delta}_4^*<0$, *then* $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_6(T_{\overline{W}})\}$ *and* $T^*=T_1^*$ *or* $T_{\overline{W}}$ *associated with the least cost.*

Proof. The proof of Theorem 6 is similar to the proof of Theorem 5. □

6. The algorithm

In this section, we shall combine above Sections 4 and 5 to outline the algorithm to help us decide the optimal cycle time and optimal order quantity.

Algorithm. *Step 1*: If $T_{\overline{W}}\ge M$, then go to step 6. Otherwise, go to Step 2.

Step 2: If $T_a<T_{\overline{W}}<M$, then go to Step 3. Otherwise, go to Step 4.

Step 3:

(1) If $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4<0$, then $T^*=T_a$ or T_4^* associated with the least cost.

(2) If $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4\ge 0$, then $T^*=T_a$ or T_3^* associated with the least cost.

(3) If $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\ge 0$, $\Delta_3<0$ and $\Delta_4<0$, then $T^*=T_a$ or T_4^* associated with the least cost.

(4) If $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\ge 0$, $\Delta_3<0$ and $\Delta_4\ge 0$, then $T^*=T_a$ or T_3^* associated with the least cost.

(5) If $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\ge 0$, $\Delta_3\ge 0$ and $\Delta_4\ge 0$, then $T^*=T_a$, T_2^* or $T_{\overline{W}}$ associated with the least cost.

(6) If $\Delta_1\ge 0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4<0$, then $T^*=T_1^*$ or T_4^* associated with the least cost.

(7) If $\Delta_1\ge 0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4\ge 0$, then $T^*=T_1^*$ or T_3^* associated with the least cost.

(8) If $\Delta_1\ge 0$, $\Delta_1^*<0$, $\Delta_2\ge 0$, $\Delta_3<0$ and $\Delta_4<0$, then $T^*=T_1^*$ or T_4^* associated with the least cost.

(9) If $\Delta_1\ge 0$, $\Delta_1^*<0$, $\Delta_2\ge 0$, $\Delta_3<0$ and $\Delta_4\ge 0$, then $T^*=T_1^*$ or T_3^* associated with the least cost.

(10) If $\Delta_1\ge 0$, $\Delta_1^*<0$, $\Delta_2\ge 0$, $\Delta_3\ge 0$ and $\Delta_4\ge 0$, then $T^*=T_1^*$, T_2^* or $T_{\overline{W}}$ associated with the least cost.

(11) If $\Delta_1\ge 0$, $\Delta_1^*\ge 0$, $\Delta_2\ge 0$, $\Delta_3<0$ and $\Delta_4<0$, then $T^*=T_1^*$ or T_4^* associated with the least cost.

(12) If $\Delta_1\ge 0$, $\Delta_1^*\ge 0$, $\Delta_2\ge 0$, $\Delta_3<0$ and $\Delta_4\ge 0$, then $T^*=T_1^*$ or T_3^* associated with the least cost.

(13) If $\Delta_1\ge 0$, $\Delta_1^*\ge 0$, $\Delta_2\ge 0$, $\Delta_3\ge 0$ and $\Delta_4\ge 0$, then $T^*=T_1^*$ or $T_{\overline{W}}$ associated with the least cost.

Step 3: If $T_{\overline{W}}<T_a<M$, then go to Step 4. Otherwise, go to Step 5.

Step 4:

(1) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3<0$, $\overline{\Delta}_3^*<0$ and $\Delta_4<0$, then $T^*=T_a$ or T_4^* associated with the least cost.

(2) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3<0$, $\overline{\Delta}_3^*<0$ and $\Delta_4\ge 0$, then $T^*=T_a$ or T_3^* associated with the least cost.

(3) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*<0$ and $\Delta_4<0$, then $T^*=T_5^*$ or T_4^* associated with the least cost.

(4) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*<0$ and $\Delta_4\ge 0$, then $T^*=T_5^*$ or T_3^* associated with the least cost.

(5) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*\ge 0$ and $\Delta_4\ge 0$, then $T^*=T_5^*$.

(6) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3<0$, $\overline{\Delta}_3^*<0$ and $\Delta_4<0$, then $T^*=T_a$ or T_4^* associated with the least cost.

(7) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3<0$, $\overline{\Delta}_3^*<0$ and $\Delta_4\ge 0$, then $T^*=T_a$ or T_3^* associated with the least cost.

(8) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*<0$ and $\Delta_4<0$, then $T^*=T_5^*$ or T_4^* associated with the least cost.

(9) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*<0$ and $\Delta_4\ge 0$, then $T^*=T_5^*$ or T_3^* associated with the least cost.

(10) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*\ge 0$ and $\Delta_4\ge 0$, then $T^*=T_5^*$.

(11) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2\ge 0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*<0$ and $\Delta_4<0$, then $T^*=T_1^*$, $T_{\overline{W}}$ or T_4^* associated with the least cost.

(12) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2\ge 0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*<0$ and $\Delta_4\ge 0$, then $T^*=T_1^*$, $T_{\overline{W}}$ or T_3^* associated with the least cost.

(13) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2\ge 0$, $\overline{\Delta}_3\ge 0$, $\overline{\Delta}_3^*\ge 0$ and $\Delta_4\ge 0$, then $T^*=T_1^*$ or $T_{\overline{W}}$ associated with the least cost.

Step 5:

(1) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\Delta_3^*<0$, $\Delta_4^*<0$ and $\overline{\Delta}_4^*<0$, then $T^*=T_a$ or T_4^* associated with the least cost.

(2) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\Delta_3^*<0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*<0$, then $T^*=T_6^*$ or T_4^* associated with the least cost.

(3) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\Delta_3^*<0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*\ge 0$, then $T^*=T_6^*$.

(4) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\Delta_3^*\ge 0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*<0$, then $T^*=T_5^*$ or T_4^* associated with the least cost.

(5) If $\overline{\Delta}_1<0$, $\overline{\Delta}_2<0$, $\Delta_3^*\ge 0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*\ge 0$, then $T^*=T_5^*$.

(6) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\Delta_3^*<0$, $\Delta_4^*<0$ and $\overline{\Delta}_4^*<0$, then $T^*=T_a$ or T_4^* associated with the least cost.

(7) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\Delta_3^*<0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*<0$, then $T^*=T_6^*$ or T_4^* associated with the least cost.

(8) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\Delta_3^*<0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*\ge 0$, then $T^*=T_6^*$.

(9) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\Delta_3^*\ge 0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*<0$, then $T^*=T_5^*$ or T_4^* associated with the least cost.

(10) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2<0$, $\Delta_3^*\ge 0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*\ge 0$, then $T^*=T_5^*$.

(11) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2\ge 0$, $\Delta_3^*\ge 0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*<0$, then $T^*=T_1^*$, $T_{\overline{W}}$ or T_4^* associated with the least cost.

(12) If $\overline{\Delta}_1\ge 0$, $\overline{\Delta}_2\ge 0$, $\Delta_3^*\ge 0$, $\Delta_4^*\ge 0$ and $\overline{\Delta}_4^*\ge 0$, then $T^*=T_1^*$ or $T_{\overline{W}}$ associated with the least cost.

Step 6: If $T_a < M < T_{\overline{W}}$ or $M < T_a < T_{\overline{W}}$, then go to Step 7. Otherwise, go to Step 8.

Step 7:

(1) If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$ and $\tilde{\Delta}_4 < 0$, then $T^* = T_a$ or T_4^* associated with the least cost.
(2) If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 < 0$, then $T^* = T_a$, T_2^* or T_4^* associated with the least cost.
(3) If $\Delta_1 < 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 \geq 0$, then $T^* = T_a$, T_2^* or $T_{\overline{W}}$ associated with the least cost.
(4) If $\Delta_1 \geq 0$, $\Delta_1^* < 0$, $\Delta_2 < 0$ and $\tilde{\Delta}_4 < 0$, then $T^* = T_1^*$ or T_4^* associated with the least cost.
(5) If $\Delta_1 \geq 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 < 0$, then $T^* = T_1^*$, T_2^* or T_4^* associated with the least cost.
(6) If $\Delta_1 \geq 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 \geq 0$, then $T^* = T_1^*$, T_2^* or $T_{\overline{W}}$ associated with the least cost.
(7) If $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 < 0$, then $T^* = T_1^*$ or T_4^* associated with the least cost.
(8) If $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 \geq 0$, then $T^* = T_1^*$ or $T_{\overline{W}}$ associated with the least cost.

Step 8:

(1) If $\overline{\Delta}_1 < 0$, $\Delta_2^* < 0$, $\Delta_4^* < 0$ and $\overline{\Delta}_4^* < 0$, then $T^* = T_a$ or T_4^* associated with the least cost.
(2) If $\overline{\Delta}_1 < 0$, $\Delta_2^* < 0$, $\Delta_4^* \geq 0$ and $\overline{\Delta}_4^* < 0$, then $T^* = T_b^*$ or T_4^* associated with the least cost.
(3) If $\overline{\Delta}_1 < 0$, $\Delta_2^* < 0$, $\Delta_4^* \geq 0$ and $\overline{\Delta}_4^* < 0$, then $T^* = T_6^*$.
(4) If $\overline{\Delta}_1 \geq 0$, $\Delta_2^* < 0$, $\Delta_4^* < 0$ and $\overline{\Delta}_4^* < 0$, then $T^* = T_1^*$, T_a or T_4^* associated with the least cost.
(5) If $\overline{\Delta}_1 \geq 0$, $\Delta_2^* < 0$, $\Delta_4^* \geq 0$ and $\overline{\Delta}_4^* < 0$, then $T^* = T_1^*$, T_6^* or T_4^* associated with the least cost.
(6) If $\overline{\Delta}_1 \geq 0$, $\Delta_2^* < 0$, $\Delta_4^* \geq 0$ and $\overline{\Delta}_4^* \geq 0$, then $T^* = T_1^*$ or T_6^* associated with the least cost.
(7) If $\overline{\Delta}_1 \geq 0$, $\Delta_2^* \geq 0$, $\Delta_4^* \geq 0$ and $\overline{\Delta}_4^* < 0$, then $T^* = T_1^*$, $T_{\overline{W}}$ or T_4^* associated with the least cost.
(8) If $\overline{\Delta}_1 \geq 0$, $\Delta_2^* \geq 0$, $\Delta_4^* \geq 0$ and $\overline{\Delta}_4^* \geq 0$, then $T^* = T_1^*$ or $T_{\overline{W}}$ associated with the least cost.

7. Numerical examples

To illustrate all results obtained in this paper, let us apply the proposed method to efficiently and quickly solve the following examples. The optimal solution for the retailer is summarized in Tables 1–5. To investigate the effects of S, λ and D on the optimal replenishment cycle time T^*, the optimal ordering quantity Q^* and the optimal total cost $TVC(T^*)$ derived from the proposed method, the following inferences can be derived based in Tables 6–10.

(1) From Tables 6–10, we find out when ordering cost (S) increases, T_a and $T_{\overline{W}}$ remain the same, both the optimal replenishment cycle time T^* and the optimal total cost $TVC(T^*)$ increase. Besides, if ordering cost (S) increases, the optimal ordering quantity Q^* increases. This implies that if ordering cost increases, the decision of renting additional warehouse is changed from "No" to "Yes". This implication that the retailer needs to rent an additional warehouse to stock the inventory or not is significantly affected by the value of S.

Table 1
The optimal ordering policy using Theorem 2.

	S	A	C	h	k	R	I	M	λ	D	$T_{\overline{W}}$	T_a	Δ_1	Δ_1^*	Δ_2	Δ_3	Δ_4	T^*	Q^*	$TVC(T^*)$	Use RW?
(1)	70	50	7	5	6	0.15	0.12	0.3	0.03	200	0.2491	0.1497	<0	<0	<0	<0	<0	0.4062	81.73	1917.1	Yes
(2)	7	50	7	5	6	0.15	0.12	0.3	0.03	3000	0.0167	0.010	<0	<0	<0	<0	>0	0.0724	217.32	21792	Yes
(3)	26	5	5	5	6	0.15	0.12	0.3	0.03	100	0.2987	0.0999	<0	<0	>0	<0	<0	0.3013	30.27	680.7805	Yes
(4)	12	6	7	5	6	0.15	0.12	0.3	0.03	80	0.2491	0.1248	<0	<0	>0	<0	>0	0.2527	20.29	677.0893	Yes
(5)	5	6	5	5	6	0.15	0.12	0.3	0.03	100	0.2491	0.0999	<0	<0	>0	>0	>0	0.0990	9.91	580.4822	No
(6)	2	10	5	1.2	1.5	0.15	0.12	0.3	0.03	27	0.2950	0.2583	>0	<0	<0	<0	<0	0.2560	6.94	150.6084	No
(7)	2	500	7	5	6	0.15	0.12	0.3	0.03	3000	0.1663	0.0999	>0	<0	<0	<0	>0	0.0140	42.00	21279	No
(8)	12	7	4.5	3	4	0.15	0.12	0.3	0.03	100	0.2987	0.2491	>0	<0	>0	<0	<0	0.2460	24.69	547.4236	No
(9)	2	290	7	5	6	0.15	0.12	0.3	0.03	3000	0.1663	0.0999	>0	<0	>0	<0	>0	0.0140	42.01	21279	No
(10)	2	200	7	5	6	0.15	0.12	0.3	0.03	3000	0.1663	0.0999	>0	<0	>0	>0	>0	0.0140	42.01	21279	No
(11)	10	2	6.5	2	2.5	0.2	0.03	0.3	0.03	95	0.2934	0.2621	>0	>0	>0	<0	<0	0.3031	28.93	688.9891	Yes
(12)	10	2	7	2	2.5	0.2	0.05	0.3	0.03	95	0.2934	0.2621	>0	>0	>0	<0	>0	0.2955	28.20	734.7054	Yes
(13)	2	15	7	5	6	0.15	0.12	0.3	0.03	3000	0.1663	0.0999	>0	>0	>0	>0	>0	0.014	42.01	21279	No

Table 2
The optimal ordering policy using Theorem 3.

	S	A	C	h	k	R	I	M	λ	D	$T_{\overline{W}}$	T_a	$\overline{\Delta}_1$	$\overline{\Delta}_2$	$\overline{\Delta}_3$	$\overline{\Delta}_3^*$	Δ_4	T^*	Q^*	$TVC(T^*)$	Use RW?
(1)	15	5	5	5	6	0.15	0.12	0.3	0.03	50	0.19949	0.2987	<0	<0	<0	<0	<0	0.2986	15	335.3963	No
(2)	10	5	7	5	6	0.15	0.12	0.3	0.03	800	0.0375	0.0624	<0	<0	<0	<0	>0	0.0624	49.95	5715	No
(3)	20	10	7	5	6	0.15	0.12	0.15	0.03	350	0.0856	0.1426	<0	<0	>0	<0	<0	0.1350	47.35	2702.1	No
(4)	10	5	7	5	6	0.15	0.12	0.3	0.03	600	0.0500	0.0832	<0	<0	>0	<0	>0	0.0730	43.83	4323	No
(5)	10	5	7	5	6	0.15	0.12	0.3	0.03	500	0.0599	0.0999	<0	<0	>0	>0	>0	0.0798	39.97	3624.3	No
(6)	2.2	2	5	2	3	0.15	0.03	0.3	0.03	20	0.2788	0.2987	>0	<0	<0	<0	<0	0.2986	6.00	113.8040	No
(7)	2.2	1	5	2	3	0.15	0.03	0.35	0.03	20	0.2788	0.2987	>0	<0	<0	<0	>0	0.2986	6.00	113.6540	No
(8)	3	5	5	2	3	0.15	0.12	0.3	0.03	50	0.1994	0.2391	>0	<0	>0	<0	<0	0.2031	10.19	270.5196	No
(9)	3	2	5	2	3	0.15	0.12	0.3	0.03	50	0.1994	0.2391	>0	<0	>0	<0	>0	0.2031	10.19	270.5196	No
(10)	3	1	5	2	3	0.15	0.12	0.3	0.03	50	0.1994	0.2391	>0	<0	>0	>0	>0	0.2031	10.19	270.5196	No
(11)	10	50	7	5	6	0.2	0.03	0.3	0.03	200	0.1497	0.2491	>0	>0	>0	<0	<0	0.1497	30.00	1510.2	No
(12)	10	35	7	5	6	0.2	0.05	0.3	0.03	200	0.1497	0.2491	>0	>0	>0	<0	>0	0.1497	30.00	1510.2	No
(13)	10	5	7	5	6	0.15	0.12	0.3	0.03	200	0.1497	0.2491	>0	>0	>0	>0	>0	0.1497	30.00	1510.2	No

Table 3
The optimal ordering policy using Theorem 4.

	S	A	C	h	k	R	I	M	λ	D	$T_{\overline{W}}$	T_a	$\overline{\Delta}_1$	$\overline{\Delta}_2$	Δ_3^*	Δ_4^*	$\overline{\Delta}_4^*$	T^*	Q^*	$TVC(T^*)$	Use RW?
(1)	16	8	5	6	7	0.15	0.12	0.2	0.03	50	0.1994	0.2987	<0	<0	<0	<0	<0	0.2987	15.00	349.3553	No
(2)	10	8	5	6	7	0.15	0.12	0.2	0.03	50	0.1994	0.2987	<0	<0	<0	>0	<0	0.2395	12.02	327.1824	No
(3)	10	5	5	6	7	0.15	0.12	0.2	0.03	50	0.1994	0.2987	<0	<0	<0	>0	>0	0.2395	12.02	327.1824	No
(4)	4.2	8	4	5	6	0.15	0.12	0.25	0.03	25	0.2391	0.2788	<0	<0	>0	>0	<0	0.2418	6.07	131.6968	No
(5)	4.2	1.2	4	5	6	0.15	0.12	0.25	0.03	25	0.2391	0.2788	<0	<0	>0	>0	>0	0.2418	6.07	131.6968	No
(6)	5.5	3	3	3.5	4	0.15	0.10	0.3	0.03	30	0.2987	0.3020	>0	<0	<0	<0	<0	0.3020	9.10	123.5918	No
(7)	5.5	3	3	3.5	4	0.15	0.12	0.3	0.03	30	0.2987	0.3020	>0	<0	<0	>0	<0	0.3004	9.05	123.3231	No
(8)	7	5	5	6	7	0.15	0.12	0.2	0.03	50	0.1994	0.2987	>0	<0	<0	>0	>0	0.2011	10.09	313.5621	No
(9)	4.22	3	4.5	5	6	0.15	0.12	0.25	0.03	25	0.2391	0.2788	>0	<0	>0	>0	<0	0.2399	6.02	144.0930	No
(10)	4.2	1.5	4.5	5	6	0.15	0.12	0.25	0.03	25	0.2391	0.2788	>0	<0	>0	>0	>0	0.2399	6.02	144.0930	No
(11)	5	2	3	3.5	4	0.15	0.12	0.3	0.03	30	0.2987	0.3020	>0	>0	>0	>0	<0	0.2987	9.00	121.6496	No
(12)	3	2	3	3.5	4	0.15	0.12	0.3	0.03	30	0.2987	0.3020	>0	>0	>0	>0	>0	0.2987	9.00	114.9530	No

Table 4
The optimal ordering policy using Theorem 5.

	S	A	C	h	k	R	I	M	λ	D	$T_{\overline{W}}$	T_a	Δ_1	Δ_1^*	Δ_2	$\tilde{\Delta}_4$	T^*	Q^*	$TVC(T^*)$	Use RW?
(1)	17	5	4	5	6	0.15	0.12	0.3	0.03	20	0.4963	0.2491	<0	<0	<0	<0	0.5734	11.57	150.4720	Yes
(2)	5	16	4	5	6	0.15	0.12	0.3	0.03	15	0.6601	0.3317	<0	<0	>0	<0	0.3317	4.99	89.6516	No
(3)	10	5	4	5	6	0.15	0.12	0.3	0.03	20	0.4963	0.2491	<0	<0	>0	>0	0.2490	4.99	134.7385	No
(4)	3	20	5	3	5	0.15	0.12	0.5	0.03	15	0.6601	0.3317	>0	<0	<0	<0	0.3130	4.72	94.1219	No
(5)	3	16	5	3	5	0.20	0.12	0.4	0.03	15	0.6601	0.3317	>0	<0	>0	<0	0.3040	4.59	94.7023	No
(6)	3	5	4	5	6	0.15	0.12	0.3	0.03	20	0.4963	0.2491	>0	<0	>0	>0	0.2260	4.54	106.5026	No
(7)	2.5	0.1	5	3	5	0.20	0.01	0.2	0.03	20	0.2640	0.2491	>0	>0	>0	<0	0.2649	5.32	118.4894	Yes
(8)	3	2	5	3	5	0.15	0.12	0.5	0.03	10	0.9853	0.4963	>0	>0	>0	>0	0.3830	3.85	65.6184	No

Table 5
The optimal ordering policy using Theorem 6.

	S	A	C	h	k	R	I	M	λ	D	$T_{\overline{W}}$	T_a	$\overline{\Delta}_1$	Δ_2^*	Δ_4^*	$\overline{\Delta}_4^*$	T^*	Q^*	$TVC(T^*)$	Use RW?
(1)	15	10	4	5	6	0.15	0.12	0.1	0.03	30	0.2987	0.3976	<0	<0	<0	<0	0.3976	12.00	190.9323	No
(2)	8	10	4	5	6	0.15	0.12	0.1	0.03	30	0.2987	0.3976	<0	<0	>0	<0	0.3017	9.09	171.2791	No
(3)	8	5	4	5	6	0.15	0.12	0.1	0.03	30	0.2987	0.3976	<0	<0	>0	>0	0.3017	9.09	171.2791	No
(4)	5.5	5	3	3.5	4	0.15	0.05	0.2	0.03	30	0.2987	0.3020	>0	<0	<0	<0	0.3020	9.10	124.8675	No
(5)	8	8	4.5	5	6	0.15	0.12	0.1	0.03	30	0.2987	0.3976	>0	<0	>0	<0	0.2991	9.01	186.5354	No
(6)	8	5	4.5	5	6	0.15	0.12	0.1	0.03	30	0.2987	0.3976	>0	<0	>0	>0	0.2991	9.01	186.5354	No
(7)	7	8	4.5	5	6	0.15	0.12	0.1	0.03	30	0.2987	0.3976	>0	>0	>0	<0	0.2987	9.00	183.1871	No
(8)	6	8	4.5	5	6	0.15	0.12	0.1	0.03	30	0.2987	0.3976	>0	>0	>0	>0	0.2987	9.00	179.8388	No

Table 6
Sensitivity analysis: The effect of changing the parameter keeping all other parameters unchanged using Theorem 2.

	$T_{\overline{W}}$	T_a	Δ_1	Δ_1^*	Δ_2	Δ_3	Δ_4	T^*	Q^*	$TVC(T^*)$	Use RW?
Parameter (S)											
30	0.2987	0.2491	<0	<0	<0	<0	<0	0.3221	32.3661	680.5939	Yes
40	0.2987	0.2491	<0	<0	<0	<0	<0	0.3632	36.5186	709.7781	Yes
50	0.2987	0.2491	<0	<0	<0	<0	<0	0.3999	40.2308	735.9870	Yes
60	0.2987	0.2491	<0	<0	<0	<0	<0	0.4336	43.6432	759.9813	Yes
Parameter (λ)											
0.03	0.2987	0.2491	<0	<0	<0	<0	<0	0.3221	32.3661	680.5939	Yes
0.10	0.2956	0.2469	<0	<0	<0	<0	<0	0.3052	30.9905	692.3525	Yes
0.17	0.2926	0.2448	<0	<0	>0	>0	>0	0.2926	30.0000	703.7609	Yes
0.24	0.2897	0.2428	<0	<0	>0	>0	>0	0.2897	30.0000	714.7228	Yes
Parameter (D)											
100	0.2987	0.2491	<0	<0	<0	<0	<0	0.3221	32.3661	680.5939	Yes
120	0.2491	0.2077	<0	<0	<0	<0	>0	0.2925	36.1625	796.5243	Yes
140	0.2136	0.1781	<0	<0	<0	<0	>0	0.2694	37.8688	911.0062	Yes
160	0.1870	0.1559	<0	<0	<0	<0	>0	0.2511	40.3236	1024.300	Yes

Base example 1: $h=5$, $k=6$, $R=0.15$, $I=0.12$, $S=30$, $A=3$, $C=5$, $\lambda=0.03$, $D=100$, $M=0.3$, $W=25$, $\overline{W}=30$.

Table 7
Sensitivity analysis: The effect of changing the parameter keeping all other parameters unchanged using Theorem 3.

	$T_{\overline{W}}$	T_a	$\overline{\Delta}_1$	$\overline{\Delta}_2$	$\overline{\Delta}_3$	$\overline{\Delta}_3^*$	Δ_4	T^*	Q^*	$TVC(T^*)$	Use RW?
Parameter (S)											
30	0.1994	0.2391	<0	<0	<0	<0	<0	0.5662	28.5518	342.2339	Yes
40	0.1994	0.2391	<0	<0	<0	<0	<0	0.6463	32.6303	358.72923	Yes
50	0.1994	0.2391	<0	<0	<0	<0	<0	0.7174	36.2588	373.3957	Yes
60	0.1994	0.2391	<0	<0	<0	<0	<0	0.7819	39.5571	386.7357	Yes
Parameter (λ)											
0.03	0.1994	0.2391	<0	<0	<0	<0	<0	0.5662	28.5518	342.2339	Yes
0.10	0.1980	0.2372	<0	<0	<0	<0	<0	0.5174	26.5510	352.4361	Yes
0.17	0.1967	0.2352	<0	<0	<0	<0	<0	0.4786	24.9304	361.9748	No
0.24	0.1953	0.2333	<0	<0	<0	<0	<0	0.4469	23.5873	370.9717	No
Parameter (D)											
50	0.1994	0.2391	<0	<0	<0	<0	<0	0.5662	28.5518	342.2339	Yes
60	0.1663	0.1994	<0	<0	<0	<0	>0	0.5157	31.1826	400.6463	Yes
70	0.1426	0.1710	<0	<0	<0	<0	<0	0.4769	33.6229	458.2643	Yes
80	0.1248	0.1497	<0	<0	<0	<0	<0	0.4458	35.9036	515.2499	Yes

Base example 2 $h=2$, $k=3$, $R=0.15$, $I=0.12$, $S=30$, $A=1$, $C=5$, $\lambda=0.03$, $D=50$, $M=0.3$, $W=12$, $\overline{W}=10$.

Table 8
Sensitivity analysis: The effect of changing the parameter keeping all other parameters unchanged using Theorem 4.

	$T_{\overline{W}}$	T_a	$\overline{\Delta}_1$	$\overline{\Delta}_2$	Δ_3^*	Δ_4^*	$\overline{\Delta}_4^*$	T^*	Q^*	$TVC(T^*)$	Use RW?
Parameter (S)											
20	0.1994	0.2987	<0	<0	<0	<0	<0	0.2987	14.9971	362.7512	No
30	0.1994	0.2987	<0	<0	<0	<0	<0	0.2987	14.9971	396.2408	No
40	0.1994	0.2987	<0	<0	<0	>0	<0	0.4831	24.3288	423.2626	Yes
50	0.1994	0.2987	<0	<0	<0	>0	<0	0.5314	26.7808	442.9785	Yes
Parameter (λ)											
0.03	0.1994	0.2987	<0	<0	<0	<0	<0	0.2987	14.9971	396.2408	No
0.10	0.1980	0.2956	<0	<0	<0	<0	<0	0.2956	14.9955	402.2986	No
0.17	0.1967	0.2926	<0	<0	<0	<0	<0	0.2926	14.9952	408.3020	No
0.24	0.1953	0.2897	<0	<0	<0	<0	<0	0.2897	14.9951	414.2927	No
Parameter (D)											
50	0.1994	0.2987	<0	<0	<0	<0	<0	0.2987	14.9971	396.2408	No
55	0.1813	0.2716	<0	<0	<0	<0	<0	0.2716	14.9990	430.5989	No
60	0.1663	0.2491	<0	<0	<0	<0	<0	0.2491	14.9999	464.9858	No
65	0.1535	0.2300	<0	<0	<0	<0	<0	0.2300	24.4715	497.1552	Yes

Base example 3 $h=6$, $k=7$, $R=0.15$, $I=0.12$, $S=30$, $A=5$, $C=5$, $\lambda=0.03$, $D=50$, $M=0.2$, $W=15$, $\overline{W}=10$.

Table 9
Sensitivity analysis: The effect of changing the parameter keeping all other parameters unchanged using Theorem 5.

	$T_{\overline{W}}$	T_a	Δ_1	Δ_1^*	Δ_2	$\tilde{\Delta}_4$	T^*	Q^*	$TVC(T^*)$	Use RW?
Parameter (S)										
20	0.4963	0.2491	<0	<0	<0	<0	0.6287	12.6931	140.4755	Yes
30	0.4963	0.2491	<0	<0	<0	<0	0.7504	15.1779	154.9769	Yes
40	0.4963	0.2491	<0	<0	<0	<0	0.8547	17.3148	167.4370	Yes
50	0.4963	0.2491	<0	<0	<0	<0	0.9472	19.2155	178.5363	Yes
Parameter (λ)										
0.03	0.4963	0.2491	<0	<0	<0	<0	0.7504	15.1779	154.9769	Yes
0.10	0.4879	0.2469	<0	<0	<0	<0	0.7059	14.6283	159.8188	Yes
0.17	0.4799	0.2448	<0	<0	<0	<0	0.6677	14.1410	164.4846	Yes
0.24	0.4722	0.2428	<0	<0	<0	<0	0.6343	13.7026	168.9949	Yes
Parameter (D)										
20	0.4963	0.2491	<0	<0	<0	<0	0.7504	15.1779	154.9769	Yes
70	0.1426	0.0714	<0	<0	<0	<0	0.3988	28.0803	421.2913	Yes
120	0.0832	0.0416	<0	<0	<0	<0	0.3053	36.8078	663.2904	Yes
170	0.0588	0.0294	<0	<0	<0	<0	0.2575	43.9397	895.9266	Yes

Base example 4 $h=3$, $k=5$, $R=0.15$, $I=0.12$, $S=30$, $A=2$, $C=4$, $\lambda=0.03$, $D=20$, $M=0.3$, $W=5$, $\overline{W}=10$.

Table 10
Sensitivity analysis: The effect of changing the parameter keeping all other parameters unchanged using Theorem 6.

	$T_{\overline{W}}$	T_a	$\overline{\Delta}_1$	Δ_2^*	Δ_4^*	$\overline{\Delta}_4^*$	T^*	Q^*	$TVC(T^*)$	Use RW?
Parameter (S)										
20	0.2987	0.3976	<0	<0	<0	<0	0.3976	11.9981	172.6925	No
30	0.2987	0.3976	<0	<0	<0	<0	0.3976	11.9981	197.8461	No
40	0.2987	0.3976	<0	<0	<0	<0	0.6857	20.7846	214.4331	Yes
50	0.2987	0.3976	<0	<0	<0	<0	0.7541	22.8814	228.3230	Yes
Parameter (λ)										
0.03	0.2987	0.3976	<0	<0	<0	<0	0.6857	20.7846	214.4331	Yes
0.10	0.2956	0.3922	<0	<0	<0	<0	0.6551	20.3113	219.7542	Yes
0.17	0.2926	0.3870	<0	<0	<0	<0	0.6276	19.8686	224.9420	Yes
0.24	0.2897	0.3819	<0	<0	<0	<0	0.6029	19.4626	230.0075	Yes
Parameter (D)										
30	0.2987	0.3976	<0	<0	<0	<0	0.6857	20.7846	214.4331	Yes
80	0.1123	0.1497	<0	<0	<0	<0	0.4145	33.3641	622.3935	Yes
130	0.0692	0.0922	<0	<0	<0	<0	0.3243	42.3621	652.9878	Yes
180	0.0500	0.0666	<0	<0	<0	<0	0.2753	49.7592	849.8061	Yes

Base example 5 $h=5$, $k=6$, $R=0.15$, $I=0.12$, $S=40$, $A=5$, $C=3$, $\lambda=0.03$, $D=30$, $M=0.1$, $W=12$, $\overline{W}=9$.

(2) From Tables 6–10, we find out when the deterioration rate (λ) increases, the optimal total cost $TVC(T^*)$ increases, however, T_a, $T_{\overline{W}}$ and the optimal replenishment cycle time T^* decrease. Besides, if the deterioration rate (λ) increases, the optimal ordering quantity Q^* decreases. This implies that if λ increases, the decision of renting additional warehouse is changed from "Yes" to "No". This implication that the retailer needs to rent an additional warehouse to stock the inventory or not is significantly affected by the value of λ.

(3) From Tables 6–10, we find out when the demand rate (D) increases, the optimal total cost $TVC(T^*)$ increases, T_a, $T_{\overline{W}}$ and the optimal replenishment cycle time T^* decrease. Likewise, if the demand rate (D) increases, the optimal ordering quantity Q^* increases. This implies that if the demand rate increases, the order quantity may exceed owned warehouse capacity, then the decision of renting additional warehouse is changed from "No" to "Yes". This implication that the retailer needs to rent an additional warehouse to stock the inventory or not is significantly affected by the value of D.

8. Summary and conclusions

This study combines the work of Chung and Liao (2004) and Chung and Huang (2006) to optimize ordering policy for a deteriorating commodity under capacity constraint when trade credit is linked to ordering quantity, a situation designed to reflect real world business situations. Using Theorems 2–6, the decision-maker can easily determine whether it will be financially advantageous to rent a warehouse to hold much more items to obtain a trade credit period. In fact, in situations involving unlimited storage space, the inventory model discussed in this study becomes identical to that considered by Chung and Liao (2004). Finally, numerical examples are used to illustrate all of the study results. From the sensitivity analysis, we can see that the ordering cost, deterioration rate and demand rate cost affect the total cost of the retailer.

Appendix A. Proof of Lemma 1

(1-1) When $T_a < T_{\overline{W}} < M$, Eqs. (7)–(9) yield

$$TVC_2(T)-TVC_3(T) \geq \frac{CRD}{2\lambda^2 T}(2e^{\lambda T}-2\lambda T-2-\lambda^2T^2)+CIDM$$

and

$$TVC_2(T)-TVC_4(T)=\frac{CRD}{2\lambda^2 T}(2e^{\lambda T}-2e^{\lambda(T-M)}-2\lambda M)+\frac{CIDM^2}{2T}$$

Fortunately, Lemmas 1 and 2 in Chung and Liao (2004) imply that $TVC_2(T) > TVC_3(T)$ if $T>0$ and $TVC_2(T) > TVC_4(T)$ if $T \geq M$, respectively. Finally, Eqs. (8) and (9) yield

$$\begin{aligned} TVC_3(T)-TVC_4(T) &= \frac{CID}{2T}(T-M)^2-\frac{CRD}{\lambda^2 T}[e^{\lambda(T-M)}-\lambda(T-M)-1] \\ &< \frac{CRD}{\lambda^2 T}\left[\frac{\lambda^2(T-M)^2}{2}-e^{\lambda(T-M)}+\lambda(T-M)+1\right] \\ &\leq 0 \quad \text{if} \quad T \geq M \end{aligned}$$

Therefore, $TVC_3(T) \leq TVC_4(T)$ if $T \geq M$. It is obvious that $TVC_1(T_a) < TVC_2(T_a)$, $TVC_2(T_{\overline{W}}) > TVC_3(T_{\overline{W}})$ and $TVC_3(M)=TVC_4(M)$. Hence, $TVC(T)$ is continuous except when $T=T_{\overline{W}}$ and $T=T_a$.

(1-2) When $T_{\overline{W}} \leq T_a < M$, Eqs. (6) and (11) yield

$$TVC_1(T)-TVC_5(T) \geq \frac{CRD}{2\lambda^2 T}(2e^{\lambda T}-2\lambda T-2-\lambda^2T^2)+CIDM$$

Likewise, Lemma 1 in Chung and Liao (2004) implies that $TVC_1(T) > TVC_5(T)$ if $T>0$. It can clearly be seen that $TVC_1(T_{\overline{W}}) > TVC_5(T_{\overline{W}})$, $TVC_5(T_a) < TVC_3(T_a)$ and $TVC_3(M)=TVC_4(M)$. Consequently, $TVC(T)$ is continuous except when $T=T_{\overline{W}}$ and $T=T_a$.

(1-3) When $T_{\overline{W}} < M < T_a$, Eqs. (11) and (13) yield

$$TVC_5(T)-TVC_6(T)=\frac{CID}{2T}(T-M)^2-\frac{CRD}{\lambda^2 T}[e^{\lambda(T-M)}-\lambda(T-M)-1]$$

As implied above, we find that $TVC_5(T) \leq TVC_6(T)$ if $T \geq M$. It is obvious that $TVC_5(M)=TVC_6(M)$, $TVC_1(T_{\overline{W}}) > TVC_5(T_{\overline{W}})$ and $TVC_6(T_a) < TVC_4(T_a)$. Consequently, $TVC(T)$ is continuous except when $T=T_{\overline{W}}$ and $T=T_a$.

Incorporating the above arguments, we have completed the proof of Lemma 1(1).

(2-1) When $T_a < M < T_{\overline{W}}$, since $TVC_2(T) > TVC_4(T)$ for $T \geq M$, it is clear that $TVC_2(T_{\overline{W}}) > TVC_4(T_{\overline{W}})$ and $TVC_1(T_a) < TVC_2(T_a)$. Consequently, $TVC(T)$ is continuous except when $T=T_{\overline{W}}$ and $T=T_a$.

(2-2) When $M<T_a<T_{\overline{W}}$, the annual total variable cost function is the same in (2-1), so $TVC(T)$ is continuous except when $T=T_{\overline{W}}$ and $T=T_a$.

(2-3) When $M<T_{\overline{W}}<T_a$, Eqs. (6) and (13) yield

$$TVC_1(T)-TVC_6(T)=\frac{CRD}{2\lambda^2 T}(2e^{\lambda T}-2e^{\lambda(T-M)}-2\lambda M)+\frac{CIDM^2}{2T}$$

As implied above, we find that $TVC_1(T)>TVC_6(T)$ if $T\geq M$. It is obvious that $TVC_1(T_{\overline{W}})>TVC_6(T_{\overline{W}})$ and $TVC_6(T_a)<TVC_4(T_a)$. Consequently, $TVC(T)$ is continuous except when $T=T_{\overline{W}}$ and $T=T_a$.

Incorporating the above arguments, we have completed the proof of Lemma 1(2).

Appendix B. Proof of Lemma 2

(1) Since $f'(T)=\lambda^2 Te^{\lambda T}>0$ if $T>0$ which implies $f(T)$ is increasing on $[0,\infty)$. From $f(0)=0$, $f(T)>f(0)=0$ for $T>0$. Consequently, $f(T)>0$ for $T>0$.

(2) Since $g'(T)=\lambda^2 Te^{\lambda(T-M)}>0$ if $T>0$ which implies $g(T)$ is increasing on $[0,\infty)$. For $g(M)=0$, it yields $0=g(M)\leq g(T)$ for $T\geq M$. Consequently, $g(T)\geq 0$ for $T\geq M$.

Incorporating the above arguments, we have completed the proof of Lemma 2.

Appendix C. Proof of Theorem 1

Since Lemma 3 in Chung et al. (2001) implies that $TVC''_1(T)$ and $TVC''_5(T)$ are positive for $T>0$. Additionally, since $f(T_a)>0$, $TVC''_2(T)$ and $TVC''_3(T)$ are also positive for $T>0$. Meanwhile, Lemma 2 in Chung et al. (2001) implies that $TVC''_4(T)$ and $TVC''_6(T)$ are positive for $T\geq M$.

Appendix D. Proof of Theorem 2

(1) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4<0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*>T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*>M$ and $T_4^*>M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
  (ii) $TVC_2(T)$ is decreasing on $[T_a,T_{\overline{W}}]$.
  (iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},M]$.
  (iv) $TVC_4(T)$ is decreasing on $[M,T_4^*]$ and increasing on $[T_4^*,\infty)$.

  In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$ and $TVC_3(M)=TVC_4(M)$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_a),TVC_4(T_4^*)\}$ and $T^*=T_a$ or T_4^* associate with the least cost.

(2) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4\geq 0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*>T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
  (ii) $TVC_2(T)$ is decreasing on $[T_a,T_{\overline{W}}]$.
  (iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},T_3^*]$ and increasing on $[T_3^*,M]$.
  (iv) $TVC_4(T)$ is increasing on $[M,\infty)$.

  In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$ and $TVC_3(M)=TVC_4(M)$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_a),TVC_3(T_3^*)\}$ and $T^*=T_a$ or T_3^* associate with the least cost.

(3) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\geq 0$, $\Delta_3<0$ and $\Delta_4<0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*>M$ and $T_4^*>M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
  (ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
  (iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},M]$.
  (iv) $TVC_4(T)$ is decreasing on $[M,T_4^*]$ and increasing on $[T_4^*,\infty)$.

  In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_a),TVC_4(T_4^*)\}$ and $T^*=T_a$ or T_4^* associate with the least cost.

(4) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\geq 0$, $\Delta_3<0$ and $\Delta_4\geq 0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
  (ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
  (iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},T_3^*]$ and increasing on $[T_3^*,M]$.
  (iv) $TVC_4(T)$ is increasing on $[M,\infty)$.

  In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_a),TVC_3(T_3^*)\}$ and $T^*=T_a$ or T_3^* associate with the least cost.

(5) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\geq 0$, $\Delta_3\geq 0$ and $\Delta_4\geq 0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*\leq T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
  (ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
  (iii) $TVC_3(T)$ is increasing on $[T_{\overline{W}},M]$.
  (iv) $TVC_4(T)$ is increasing on $[M,\infty)$.

  In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_a),\ TVC_2(T_2^*),\ TVC_3(T_{\overline{W}})\}$ and $T^*=T_a$, T_2^* or $T_{\overline{W}}$ associate with the least cost.

(6) $\Delta_1\geq 0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4<0$ imply that $T_1^*\leq T_a$, $T_2^*>T_a$, $T_2^*>T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*>M$ and $T_4^*>M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
  (ii) $TVC_2(T)$ is decreasing on $[T_a,T_{\overline{W}}]$.
  (iii) $TVC_3(T)$ is increasing on $[T_{\overline{W}},M]$.
  (iv) $TVC_4(T)$ is decreasing on $[M,T_4^*]$ and increasing on $[T_4^*,\infty)$.

  In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_4(T_4^*)\}$ and $T^*=T_1^*$ or T_4^* associate with the least cost.

(7) $\Delta_1\geq 0$, $\Delta_1^*<0$, $\Delta_2<0$, $\Delta_3<0$ and $\Delta_4\geq 0$ imply that $T_1^*\leq T_a$, $T_2^*>T_a$, $T_2^*>T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
  (ii) $TVC_2(T)$ is decreasing on $[T_a,T_{\overline{W}}]$.
  (iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},T_3^*]$ and increasing on $[T_3^*,M]$.
  (iv) $TVC_4(T)$ is decreasing on $[M,T_4^*]$ and increasing on $[T_4^*,\infty)$.

  In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_3(T_3^*)\}$ and $T^*=T_1^*$ or T_3^* associate with the least cost.

(8) $\Delta_1\geq 0$, $\Delta_1^*<0$, $\Delta_2\geq 0$, $\Delta_3<0$ and $\Delta_4<0$ imply that $T_1^*\leq T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*>M$ and $T_4^*>M$. Furthermore, Theorem 1 yields
  (i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.

(ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
(iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},M]$.
(iv) $TVC_4(T)$ is decreasing on $[M,T_4^*]$ and increasing on $[T_4^*,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_4(T_4^*)\}$ and $T^*=T_1^*$ or T_4^* associate with the least cost.

(9) $\Delta_1\geq 0$, $\Delta_1^*<0$, $\Delta_2\geq 0$, $\Delta_3<0$ and $\Delta_4\geq 0$ imply that $T_1^*\leq T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
(ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
(iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},T_3^*]$ and increasing on $[T_3^*,M]$.
(iv) $TVC_4(T)$ is increasing on $[M,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that$TVC(T^*)=\min\{TVC_1(T_1^*),TVC_3(T_3^*)\}$ and $T^*=T_1^*$ or T_3^* associate with the least cost.

(10) $\Delta_1\geq 0$, $\Delta_1^*<0$, $\Delta_2\geq 0$, $\Delta_3\geq 0$ and $\Delta_4\geq 0$ imply that $T_1^*\leq T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*\leq T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
(ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
(iii) $TVC_3(T)$ is increasing on $[T_{\overline{W}},M]$.
(iv) $TVC_4(T)$ is increasing on $[M,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$ and $TVC_2(T)>TVC_3(T)$ for $T>0$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),\ TVC_2(T_2^*),\ TVC_3(T_{\overline{W}})\}$ and $T^*=T_1^*$, T_2^* or $T_{\overline{W}}$ associate with the least cost.

(11) $\Delta_1\geq 0$, $\Delta_1^*\geq 0$, $\Delta_2\geq 0$, $\Delta_3<0$ and $\Delta_4<0$ imply that $T_1^*\leq T_a$, $T_2^*\leq T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*>M$ and $T_4^*>M$. Furthermore, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
(ii) $TVC_2(T)$ is increasing on $[T_a,T_{\overline{W}}]$.
(iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},M]$.
(iv) $TVC_4(T)$ is decreasing on $[M,T_4^*]$ and increasing on $[T_4^*,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$. Combining (i), (ii), (iii) and (iv), we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_4(T_4^*)\}$ and T^* is T_1^* or T_4^* associated with the least cost.

(12) $\Delta_1\geq 0$, $\Delta_1^*\geq 0$, $\Delta_2\geq 0$, $\Delta_3<0$ and $\Delta_4\geq 0$ imply that $T_1^*\leq T_a$, $T_2^*\leq T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*>T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
(ii) $TVC_2(T)$ is increasing on $[T_a,T_{\overline{W}}]$.
(iii) $TVC_3(T)$ is decreasing on $[T_{\overline{W}},T_3^*]$ and increasing on $[T_3^*,M]$.
(iv) $TVC_4(T)$ is increasing on $[M,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$. Combining (i), (ii), (iii) and (iv), we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_3(T_3^*)\}$ and T^* is T_1^* or T_3^* associated with the least cost.

(13) $\Delta_1\geq 0$, $\Delta_1^*\geq 0$, $\Delta_2\geq 0$, $\Delta_3\geq 0$ and $\Delta_4\geq 0$ imply that $T_1^*\leq T_a$, $T_2^*\leq T_a$, $T_2^*\leq T_{\overline{W}}$, $T_3^*\leq T_{\overline{W}}$, $T_3^*\leq M$ and $T_4^*\leq M$. Furthermore, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
(ii) $TVC_2(T)$ is increasing on $[T_a,T_{\overline{W}}]$.
(iii) $TVC_3(T)$ is increasing on $[T_{\overline{W}},M]$.
(iv) $TVC_4(T)$ is increasing on $[M,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$, $TVC_2(T_{\overline{W}})>TVC_3(T_{\overline{W}})$, $TVC_3(M)=TVC_4(M)$. Combining (i), (ii), (iii) and (iv), we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_3(T_{\overline{W}})\}$ and T^* is T_1^* or $T_{\overline{W}}$ associated with the least cost.

Appendix E. Proof of Theorem 5

(1) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2<0$ and $\tilde{\Delta}_4<0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*>T_{\overline{W}}$ and $T_4^*>T_{\overline{W}}$. Then, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
(ii) $TVC_2(T)$ is decreasing on $[T_a,T_{\overline{W}}]$.
(iii) $TVC_4(T)$ is decreasing on $[T_{\overline{W}},T_4^*]$ and increasing on $[T_4^*,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$ and $TVC_2(T_{\overline{W}})>TVC_4(T_{\overline{W}})$. As shown above, we obtain that $TVC(T^*)=\min\{TVC_1(T_a),TVC_4(T_4^*)\}$. Hence, $T^*=T_a$ or T_4^* associated with the least cost.

(2) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\geq 0$ and $\tilde{\Delta}_4<0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$ and $T_4^*>T_{\overline{W}}$. Then, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
(ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
(iii) $TVC_4(T)$ is decreasing on $[T_{\overline{W}},T_4^*]$ and increasing on $[T_4^*,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$ and $TVC_2(T_{\overline{W}})>TVC_4(T_{\overline{W}})$. Combining (i), (ii) and (iii), we obtain that $TVC(T^*)=\min\{TVC_1(T_a),TVC_2(T_2^*),TVC_4(T_4^*)\}$. Hence, $T^*=T_a$, T_2^* or T_4^* associated with the least cost.

(3) $\Delta_1<0$, $\Delta_1^*<0$, $\Delta_2\geq 0$ and $\tilde{\Delta}_4\geq 0$ imply that $T_1^*>T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$ and $T_4^*\leq T_{\overline{W}}$. Then, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_a]$.
(ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
(iii) $TVC_4(T)$ is increasing on $[T_{\overline{W}},\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$ and $TVC_2(T_{\overline{W}})>TVC_4(T_{\overline{W}})$. Combining (i), (ii) and (iii), we obtain that $TVC(T^*)=\min\{TVC_1(T_a),TVC_2(T_2^*),TVC_4(T_{\overline{W}})\}$. Hence, $T^*=T_a$, T_2^* or $T_{\overline{W}}$ associated with the least cost.

(4) $\Delta_1\geq 0$, $\Delta_1^*<0$, $\Delta_2<0$ and $\tilde{\Delta}_4<0$ imply that $T_1^*\leq T_a$, $T_2^*>T_a$, $T_2^*>T_{\overline{W}}$ and $T_4^*>T_{\overline{W}}$. Then, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
(ii) $TVC_2(T)$ is increasing on $[T_a,T_{\overline{W}}]$.
(iii) $TVC_4(T)$ is decreasing on $[T_{\overline{W}},T_4^*]$ and increasing on $[T_4^*,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$ and $TVC_2(T_{\overline{W}})>TVC_4(T_{\overline{W}})$. Combining (i), (ii) and (iii), we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_4(T_4^*)\}$. Hence, $T^*=T_1^*$ or T_4^* associated with the least cost.

(5) $\Delta_1\geq 0$, $\Delta_1^*<0$, $\Delta_2\geq 0$ and $\tilde{\Delta}_4<0$ imply that $T_1^*\leq T_a$, $T_2^*>T_a$, $T_2^*\leq T_{\overline{W}}$ and $T_4^*>T_{\overline{W}}$. Then, Theorem 1 yields
(i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
(ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
(iii) $TVC_4(T)$ is decreasing on $[T_{\overline{W}},T_4^*]$ and increasing on $[T_4^*,\infty)$.

In addition, $TVC_1(T_a)<TVC_2(T_a)$ and $TVC_2(T_{\overline{W}})>TVC_4(T_{\overline{W}})$. Combining (i), (ii) and (iii), we obtain that $TVC(T^*)=\min\{TVC_1(T_1^*),TVC_2(T_2^*),TVC_4(T_4^*)\}$. Hence, $T^*=T_1^*$, T_2^* or T_4^* associated with the least cost.

(6) $\Delta_1 \geq 0$, $\Delta_1^* < 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 \geq 0$ imply that $T_1^* \leq T_a$, $T_2^* > T_a$, $T_2^* \leq T_{\overline{W}}$ and $T_4^* \leq T_{\overline{W}}$. Then, Theorem 1 yields
   (i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
   (ii) $TVC_2(T)$ is decreasing on $[T_a,T_2^*]$ and increasing on $[T_2^*,T_{\overline{W}}]$.
   (iii) $TVC_4(T)$ is increasing on $[T_{\overline{W}},\infty)$.

   In addition, $TVC_1(T_a) < TVC_2(T_a)$ and $TVC_2(T_{\overline{W}}) > TVC_4(T_{\overline{W}})$. Combining (i), (ii) and (iii), we obtain that $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_2(T_2^*), TVC_4(T_{\overline{W}})\}$. Hence, $T^* = T_1^*$, T_2^* or $T_{\overline{W}}$ associated with the least cost.

(7) $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 < 0$ imply that $T_1^* \leq T_a$, $T_2^* \leq T_a$, $T_2^* \leq T_{\overline{W}}$ and $T_4^* > T_{\overline{W}}$. Then, Theorem 1 yields
   (i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
   (ii) $TVC_2(T)$ is increasing on $[T_a,T_{\overline{W}}]$.
   (iii) $TVC_4(T)$ is decreasing on $[T_{\overline{W}},T_4^*]$ and increasing on $[T_4^*,\infty)$.

   In addition, $TVC_1(T_a) < TVC_2(T_a)$ and $TVC_2(T_{\overline{W}}) > TVC_4(T_{\overline{W}})$. Combining (i), (ii) and (iii), we obtain that $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_4^*)\}$. Hence, $T^* = T_1^*$ or T_4^* associated with the least cost.

(8) $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$, $\Delta_2 \geq 0$ and $\tilde{\Delta}_4 \geq 0$ imply that $T_1^* \leq T_a$, $T_2^* \leq T_a$, $T_2^* \leq T_{\overline{W}}$ and $T_4^* \leq T_{\overline{W}}$. Then, Theorem 1 yields
   (i) $TVC_1(T)$ is decreasing on $(0,T_1^*]$ and increasing on $[T_1^*,T_a]$.
   (ii) $TVC_2(T)$ is increasing on $[T_a,T_{\overline{W}}]$.
   (iii) $TVC_4(T)$ is increasing on $[T_{\overline{W}},\infty)$.

   In addition, $TVC_1(T_a) < TVC_2(T_a)$ and $TVC_2(T_{\overline{W}}) > TVC_4(T_{\overline{W}})$. Combining (i), (ii) and (iii), we obtain that $TVC(T^*) = \min\{TVC_1(T_1^*), TVC_4(T_{\overline{W}})\}$. Hence, $T^* = T_1^*$ or $T_{\overline{W}}$ associated with the least cost.

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